7.1 Integral as Net Change

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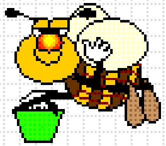
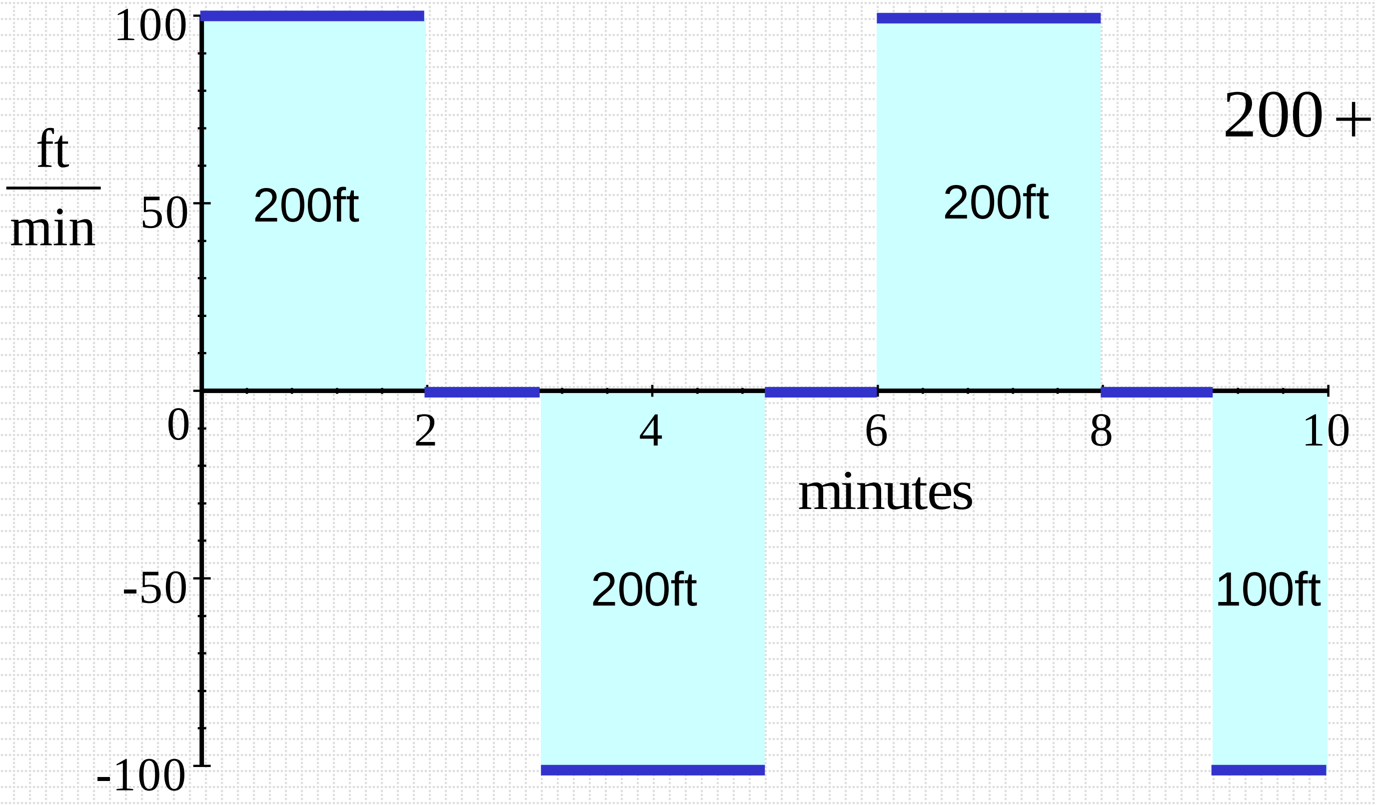
A honey bee makes several trips from the hive to a flower garden. The velocity graph is shown below.



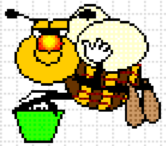
What is the total distance traveled by the bee?



$$200 + 200 + 200 + 100 = 700$$

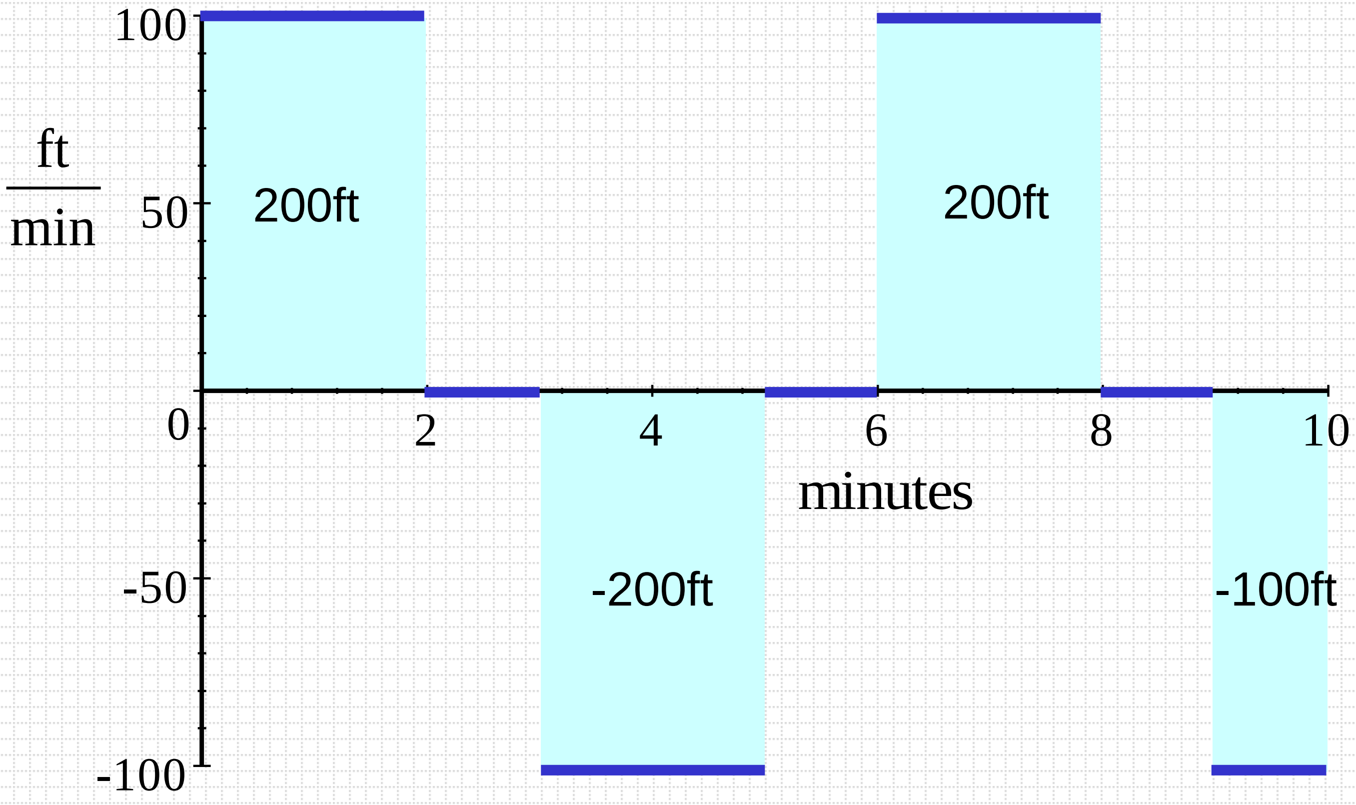
→

What is the displacement of the bee?



$$200 - 200 + 200 - 100 = 100$$

100 feet towards the hive



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To find the displacement (position shift) from the velocity function, we just integrate the function. The negative areas below the x-axis subtract from the total displacement.

$$\boxed{\text{Displacement} = \int_{t_1}^{t_2} v(t)\,dt}$$

To find distance traveled we have to use absolute value.

$$\boxed{\text{Distance Traveled} = \int_{t_1}^{t_2} \left|v(t)\right|\,dt}$$

Find the roots of the velocity equation and integrate in pieces, just like when we found the area between a curve and the x-axis. (Take the absolute value of each integral.)

Or you can use your calculator to integrate the absolute value of the velocity function.

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Recall for Rectilinear Motion (motion along a line):

Position function = $x(t)$

Velocity function = $v(t)$ $= x'(t)$

$v(t) > 0 \Rightarrow$ moving in positive direction

$v(t) < 0 \Rightarrow$ moving in negative direction

$v(t) = 0 \Rightarrow$ object stopped/changing direction (?)

$|v(t)|$ = Speed

Acceleration function = $a(t)$ $= v'(t) = x''(t)$

If $a(t)$ and $v(t)$ have the same sign $\Rightarrow$ speeding up

If $a(t)$ and $v(t)$ have opp. signs $\Rightarrow$ slowing down

→

Displacement: $\Delta x = x(t_2) - x(t_1)$

From integration: $\int_{t_1}^{t_2} v(t)dt = x(t)\Big|_{t_1}^{t_2} = x(t_2) - x(t_1)$ $= \Delta x$

$$\boxed{\text{Displacement} = \int_{t_1}^{t_2} v(t)dt}$$

A variation of the displacement formula:

$$\boxed{x(t_2) = x(t_1) + \int_{t_1}^{t_2} v(t)dt}$$

$v(t_2) = v(t_1) + \int_{t_1}^{t_2} a(t)dt$

$f(x_2) = f(x_1) + \int_{x_1}^{x_2} f'(x)dx$

Final pos. = initial position + displacement

→

Total Distance Traveled: must account for positive and negative velocities

$$\text{Total Distance} = \int_{t_1}^{t_2} \left| v(t) \right| dt$$

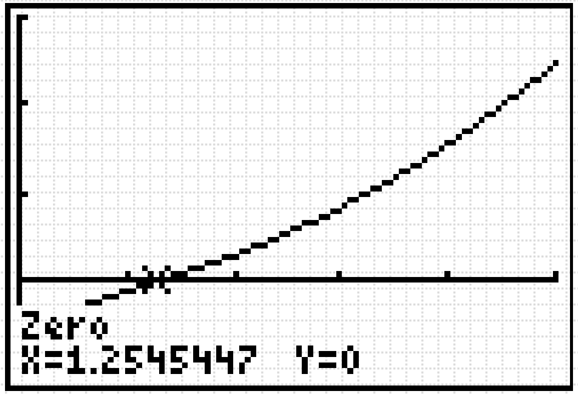
→

Example: Suppose that $v(t) = t^2 - \dfrac{8}{(t+1)^2}$ cm/s

a. Describe the motion of the object for the first five seconds. Support your answer graphically.

$v(t) = 0$ AT $t = 1.2545$

$v(t)$



$v(t) > 0$ on $(1.2545, 5] \Rightarrow$ OBJECT MOVING RT $[1.2545, 5]$

$v(t) < 0$ on $[0, 1.2545) \Rightarrow$ OBJECT MOVES LEFT $[0, 1.2545]$

$v(t)$ INCR ON $[0,5] \Rightarrow a(t) > 0$ $[0,5]$.

OBJECT SPEEDING UP $(1.2545, 5)$ SINCE $v(t) > 0$ AND $a(t) > 0$.

OBJECT SLOWING DOWN $(0, 1.2545)$ SINCE $v(t)$ AND $a(t)$ ARE OPPOSITE SIGNS.

→

Example: Suppose that $v(t) = t^2 - \dfrac{8}{(t+1)^2}$ cm/s

b. If the initial position of the object is 9, what is the position of the object at 5 seconds?

c. How far did the object travel over the first five seconds?

b) $x(5) = x(0) + \int_0^5 v(t)\,dt = 9 + \int_0^5 v(t)\,dt = 44$ cm

c) $\int_0^5 |v(t)|\,dt = 42.587$ cm

→

Example: Suppose that a car with initial velocity 5 mph has $a(t) = 2.4t$ mph/s.

How fast is the car going at the end of 8 seconds?

$$v(8) = v(0) + \int_0^8 a(t)\,dt$$

$$= 5 + 1.2t^2\Big]_0^8$$

$$= 5 + 1.2(64)$$

$$= 5 + 76.8 = \boxed{81.8 \text{ mph}}$$

→

2018 AB2

2. A particle moves along the x-axis with velocity given by $v(t) = \dfrac{10\sin\left(0.4t^2\right)}{t^2 - t + 3}$ for time $0 \le t \le 3.5$.

The particle is at position $x = -5$ at time $t = 0$.

(a) Find the acceleration of the particle at time $t = 3$.

$\cancel{a(3)} = v'(3) \approx -2.118$

$\cancel{a(3) =}$

→

2018 AB2

2. A particle moves along the x-axis with velocity given by $v(t) = \dfrac{10\sin(0.4t^2)}{t^2 - t + 3}$ for time $0 \le t \le 3.5$.

The particle is at position $x = -5$ at time $t = 0$.

(b) Find the position of the particle at time $t = 3$.

$$x(3) = \underset{x(0)}{-5} + \int_0^3 v(t)\,dt = \boxed{-1.760}2547811350\underline{1}\ldots$$

→

2018 AB2

2. A particle moves along the x-axis with velocity given by $v(t) = \dfrac{10\sin(0.4t^2)}{t^2 - t + 3}$ for time $0 \le t \le 3.5$.

The particle is at position $x = -5$ at time $t = 0$.

(c) Evaluate $\int_0^{3.5} v(t)\,dt$, and evaluate $\int_0^{3.5} |v(t)|\,dt$. Interpret the meaning of each integral in the context of the problem.

$\int_0^{3.5} v(t)\,dt = 2.844$ THE DISPLACEMENT OF PARTICLE FROM $t=0$ TO $t=3.5$

$\int_0^{3.5} |v(t)|\,dt = 3.737$ TOTAL DIST. TRAVELED BY PARTICLE FROM $t=0$ TO $t=3.5$

→

2018 AB2

2. A particle moves along the x-axis with velocity given by $v(t) = \dfrac{10\sin\left(0.4t^2\right)}{t^2 - t + 3}$ for time $0 \le t \le 3.5$.

The particle is at position $x = -5$ at time $t = 0$.

(d) A second particle moves along the x-axis with position given by $x_2(t) = t^2 - t$ for $0 \le t \le 3.5$. At what time t are the two particles moving with the same velocity?

$x_2'(t) = \boxed{2t - 1 = v(t)} \Rightarrow t = 1.570$

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Classwork:

Chapter 7 AP Packet #2, 4, 6, 10, 18

Homework:

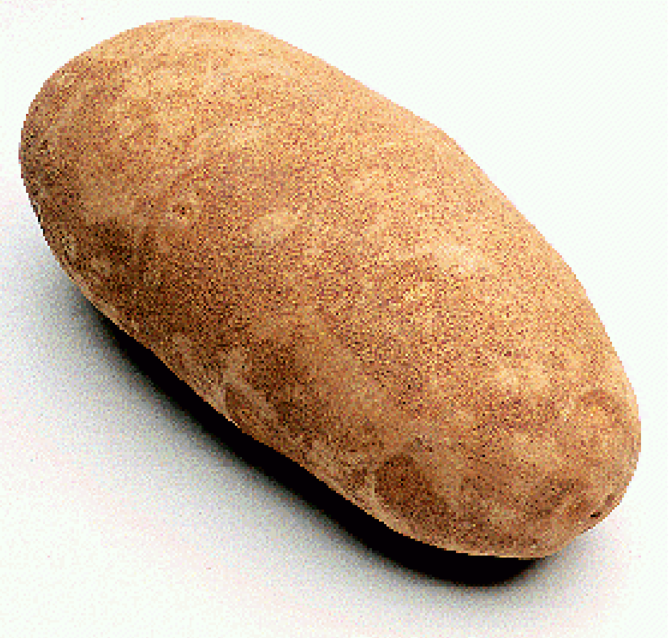
Chapter 7 AP Packet #1, 3, 5, 13, 17, 21

Example: National Potato Consumption

The rate of potato consumption for a particular country was:

$$C(t) = 2.2 + 1.1^t$$

where t is the number of years since 1970 and C is in millions of bushels per year.



The Russet Burbank

Find the amount of potatoes consumed from the beginning of 1972 to the end of 1973.

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Example 5: National Potato Consumption

$$C(t) = 2.2 + 1.1^t$$

To find the cumulative effect over time – Integrate it!

From the beginning of 1972 to the end of 1973:

$$\int_2^4 2.2 + 1.1^t \, dt = 2.2t + \frac{1}{\ln 1.1} 1.1^t \Big|_2^4 \approx 7.066$$ million bushels

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Net Change from Data: what if we don’t have the a function to work with?

Example (p. 369): A pump connected to a generator operates at a varying rate, depending on how much power is being drawn from the generator. The rate (gallons per minute) at which the pump operates is recorded at 5-minute intervals for an hour as shown in the table. How many gallons were pumped during the hour?

Time (min)	Rate (gal/min)
0	58
5	60
10	65
15	64
20	58
25	57
30	55
35	55
40	59
45	60
50	60
55	63
60	63

→

Time (min)	Rate (gal/min)
0	58
5	60
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15	64
20	58
25	57
30	55
35	55
40	59
45	60
50	60
55	63
60	63

Gallons pumped = $\int_0^{60} R(t)\,dt$

We don't have a formula for $R(t)$, so we have to approximate the integral – the trapezoidal rule works well:

$$\frac{1}{2}\cdot 5\cdot\left[58+2(60)+\cdots+2(63)+63\right]$$

$= 3582.5$ gallons

→

2010 BC1

There is no snow on Janet's driveway when snow begins to fall at midnight. From midnight to 9 A.M., snow accumulates on the driveway at a rate modeled by $f(t) = 7te^{\cos t}$ cubic feet per hour, where t is measured in hours since midnight. Janet starts removing snow at 6 A.M. $(t = 6)$. The rate $g(t)$, in cubic feet per hour, at which Janet removes snow from the driveway at time t hours after midnight is modeled by

$$g(t) = \begin{cases} 0 & \text{for } 0 \le t < 6 \\ 125 & \text{for } 6 \le t < 7 \\ 108 & \text{for } 7 \le t \le 9. \end{cases}$$

(a) How many cubic feet of snow have accumulated on the driveway by 6 A.M.?

2010 BC1

There is no snow on Janet's driveway when snow begins to fall at midnight. From midnight to 9 A.M., snow accumulates on the driveway at a rate modeled by $f(t) = 7te^{\cos t}$ cubic feet per hour, where t is measured in hours since midnight. Janet starts removing snow at 6 A.M. $(t = 6)$. The rate $g(t)$, in cubic feet per hour, at which Janet removes snow from the driveway at time t hours after midnight is modeled by

$$g(t) = \begin{cases} 0 & \text{for } 0 \le t < 6 \\ 125 & \text{for } 6 \le t < 7 \\ 108 & \text{for } 7 \le t \le 9. \end{cases}$$

(b) Find the rate of change of the volume of snow on the driveway at 8 A.M.

2010 BC1

There is no snow on Janet's driveway when snow begins to fall at midnight. From midnight to 9 A.M., snow accumulates on the driveway at a rate modeled by $f(t) = 7te^{\cos t}$ cubic feet per hour, where t is measured in hours since midnight. Janet starts removing snow at 6 A.M. $(t = 6)$. The rate $g(t)$, in cubic feet per hour, at which Janet removes snow from the driveway at time t hours after midnight is modeled by

$$g(t) = \begin{cases} 0 & \text{for } 0 \le t < 6 \\ 125 & \text{for } 6 \le t < 7 \\ 108 & \text{for } 7 \le t \le 9. \end{cases}$$

(c) Let $h(t)$ represent the total amount of snow, in cubic feet, that Janet has removed from the driveway at time t hours after midnight. Express h as a piecewise-defined function with domain $0 \le t \le 9$.

2010 BC1

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$$g(t) = \begin{cases} 0 & \text{for } 0 \le t < 6 \\ 125 & \text{for } 6 \le t < 7 \\ 108 & \text{for } 7 \le t \le 9. \end{cases}$$

(d) How many cubic feet of snow are on the driveway at 9 A.M.?

Classwork:

Chapter 7 AP Packet #23 – 25

Homework:

Chapter 7 AP Packet #27 – 29